Multiple Choice

1 C) $\cos\frac{23\pi}{12}=\cos\frac{\pi}{12}$,

$\therefore \cos^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)=\cos^{-1}\left(\cos\frac{\pi}{12}\right)=\frac{\pi}{12}$

2 A) $f(|x|)$ retains the graph of $f(x)$ for $x\geq 0$ and flips it about the y-axis.

3 D) The degree of the divisor must be higher than the degree of the remainder, $\therefore$ it could be 2 or higher.

4 A) Only the graph in (A) satisfies the positive x-intercept of $f(x)+g(x)$.

5 B) $t=x-2, \therefore y=-2(x-2)^2+3. \therefore$ Its graph is an upside down parabola with maximum at (2,3).

6 B) The projections on $\underset{\sim}{i}+\underset{\sim}{j}, -\underset{\sim}{i}+\underset{\sim}{j}, -\underset{\sim}{i}-\underset{\sim}{j}$ and $\underset{\sim}{i}-\underset{\sim}{j}$ must be $A\underset{\sim}{i}+A\underset{\sim}{j},\ -B\underset{\sim}{i}+B\underset{\sim}{j}, -A\underset{\sim}{i}-A\underset{\sim}{j}$ and $B\underset{\sim}{i}-B\underset{\sim}{j}$ respectively. Only (B) satisfies this.

7 C) Number of triangles = number of combinations using 3 points − number of 3 collinear points

$=\binom{12}{3}-\binom{3}{3}-\binom{4}{3}-\binom{5}{3}=220-1-4-10=205.$

8 D) $|\underset{\sim}{a}+\underset{\sim}{b}|<1 \Rightarrow |\underset{\sim}{a}+\underset{\sim}{b}|^2<1.$

$\therefore |\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta<1$

Since $|\underset{\sim}{a}|=|\underset{\sim}{b}|=1,\ 2+2\cos\theta<1,$

$\therefore \cos\theta<-\frac{1}{2}.$

$\therefore \frac{2\pi}{3}<\theta<\frac{4\pi}{3}$. Only (D) satisfies this range.

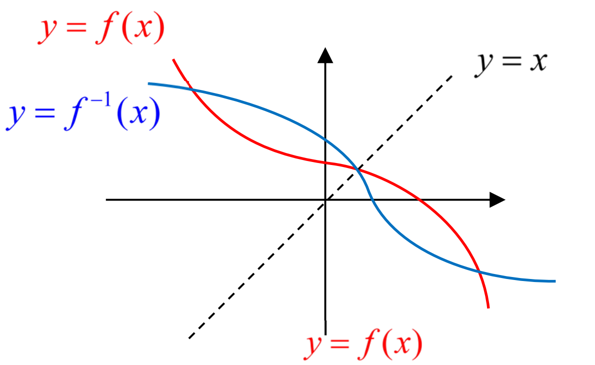
9 D) The graphs of $f(x)=0.3-x^3$ and $f^{-1}(x)=\sqrt[3]{0.3-x}$ prove that (A) and (B) are wrong as points of intersection can be on or off the line $y=x$.

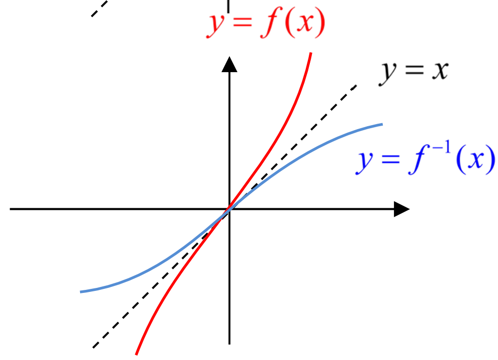
Tangents to the 2 graphs at points of intersection are parallel if the 2 graphs touch each other at the line $y=x$, e.g. $f(x)=\frac{e^x-e^{-x}}{2}, f^{-1}(x)=\ln\left(x+\sqrt{x^2+1}\right)$.

$\therefore$ (C) is wrong.

Note: The above curves are chosen so that their derivatives exist for all real numbers x.

$\therefore$ By elimination, (D) is correct.





Note: The diagrams above are displayed to illustrate my points in this question only.

10 B) $\int\frac{dy}{1+\sin y}=\int dx.$

$1+\sin y=1+\cos\left(\frac{\pi}{2}-y\right)=2\cos^2\left(\frac{\pi}{4}-\frac{y}{2}\right).$

$\therefore x=\frac{1}{2}\int\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)dy$

$=-\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)+C.$

$\therefore \tan\left(\frac{\pi}{4}-\frac{y}{2}\right)=-x+C$

$\therefore \frac{\pi}{4}-\frac{y}{2}=\tan^{-1}(-x+C).$

$\therefore y=2\left(\frac{\pi}{4}-\tan^{-1}(-x+C)\right)$

$=\frac{\pi}{2}+2\tan^{-1}(x-C).$

Question 11

(a) (i) $\underset{\sim}{u}+3\underset{\sim}{v}=\left(\underset{\sim}{i}-\underset{\sim}{j}\right)+3\left(2\underset{\sim}{i}+\underset{\sim}{j}\right)=7\underset{\sim}{i}+2\underset{\sim}{j}$.

(ii) $\underset{\sim}{u}.\underset{\sim}{v}=\left(\underset{\sim}{i}-\underset{\sim}{j}\right).\left(2\underset{\sim}{i}+\underset{\sim}{j}\right)=2-1=1$.

(b) Let $u=x^2+4, du=2xdx$.

When $x=0, u=4$; when $x=1, u=5$.

$$\int_0^1 \frac{x}{\sqrt{x^2+4}}dx=\int_4^5 \frac{du}{2\sqrt{u}}$$
$$=\left[\sqrt{u}\right]_4^5$$
$$=\sqrt{5}-2.$$

(c) The coefficient of x^r is $\binom{8}{r}\left(-\frac{1}{2}\right)^r$.

$\therefore$ The coefficient of x^2 is $\frac{1}{4}\binom{8}{2}=\frac{28}{4}=7$

and the coefficient of x^3 is $-\frac{1}{8}\binom{8}{3}=\frac{-56}{8}=-7$.

(d) $\underset{\sim}{u}.\underset{\sim}{v}=0, \therefore \begin{pmatrix} a \\ 2 \end{pmatrix}.\begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}=0$.

$a(a-7)+2(4a-1)=0$.

$a^2+a-2=0$.

$(a+2)(a-1)=0$.

$\therefore=-2$ or 1.

(e) $\sqrt{3}\sin x-3\cos x=\sqrt{3}\left(\sin x-\sqrt{3}\cos x\right)$
$$=2\sqrt{3}\left(\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x\right)$$
$$=2\sqrt{3}\left(\sin x\cos\frac{\pi}{3}-\cos x\sin\frac{\pi}{3}\right)$$
$$=2\sqrt{3}\sin\left(x-\frac{\pi}{3}\right).$$

(f) $\frac{x}{2-x}\geq 5$

$x(2-x)\geq 5(2-x)^2, x\neq 2$

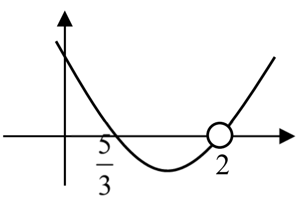
$2x-x^2\geq 20-20x+5x^2$

$6x^2-22x+20\leq 0$

$3x^2-11x+10\leq 0$

$(3x-5)(x-2)\leq 0$

$\frac{5}{3}\leq x<2$.

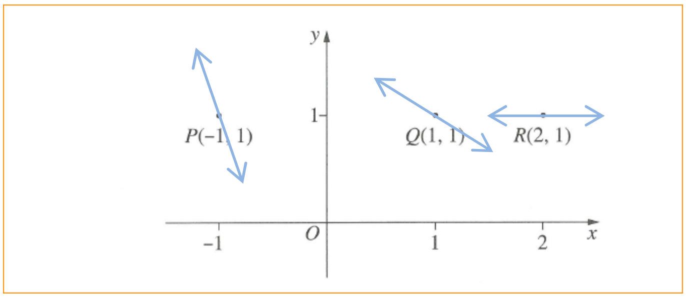


Question 12

(a) At $P(-1,1).\frac{dy}{dx}=\frac{-1-2}{1+1}=-\frac{3}{2}$.

At $Q(1,1).\frac{dy}{dx}=\frac{1-2}{1+1}=-\frac{1}{2}$.

At $R(2,1).\frac{dy}{dx}=\frac{2-2}{4+1}=0$.



(b) $41=13\times 3+2, \therefore$ by the Pigeonhole Principle, at least 1 team has at least 4 players above the age limit.

$\therefore$ At least 1 team will be penalised.

(c) $y=x\tan^{-1}x$

$y'=\tan^{-1}x+\frac{x}{1+x^2}$.

At $\left(1,\frac{\pi}{4}\right), m=\frac{\pi}{4}+\frac{1}{2}$.

Equation of the tangent is

$y-\frac{\pi}{4}=\left(\frac{\pi}{4}+\frac{1}{2}\right)(x-1)$

$y=\left(\frac{\pi}{4}+\frac{1}{2}\right)x-\frac{1}{2}$.

(d) (i) $\frac{dT}{T-T_1}=kdt$.

$\int\frac{dT}{T-T_1}=k\int dt$

$\ln\left(T-T_1\right)=kt+C.$ (1)

When $t\to\infty, T=12°\Leftrightarrow T_1=12$

When $t=0, T=92°\Leftrightarrow \ln 80=C$

When $t=5, T=76°\Leftrightarrow \ln 64=5t+\ln 80$

$\therefore 5t=\ln\frac{64}{80}=\ln\frac{4}{5}$.

$\therefore t=\frac{1}{5}\ln\frac{4}{5}$.

$\therefore$ (1) becomes $\ln\left(T-12\right)=\frac{1}{5}\ln\frac{4}{5}t+\ln 80$.

$\ln\frac{T-12}{80}=\frac{1}{5}\ln\frac{4}{5}t$

$\therefore T=12+80e^{\left(\frac{1}{5}\ln\frac{4}{5}\right)t}$.

(ii) $57 = 12 + 80e^{\left(\frac{1}{5}\ln\frac{4}{5}\right)t}$.

$45 = 80e^{\left(\frac{1}{5}\ln\frac{4}{5}\right)t}$.

$e^{\left(\frac{1}{5}\ln\frac{4}{5}\right)t} = \frac{45}{80} = \frac{9}{16}$.

$\left(\frac{1}{5}\ln\frac{4}{5}\right)t = \ln\frac{9}{16}$

$\therefore t = 5\frac{\ln\frac{9}{16}}{\ln\frac{4}{5}} \simeq 13$ minutes.

(e) The expected score of selecting 1 ball is

$\frac{3}{10}\times 10 + \frac{7}{10}\times(-5) = \frac{-5}{10} = -0.5.$

$\therefore$ When 4 balls are selected, the expected score is

$4\times(-0.5) = -2.$

(f) Let $n = 0, 15^0 + 6^1 = 7,$ which is divisible by 7. $\therefore$ true for $n = 0$.

Assume $\exists n$ so that $15^n + 6^{2n+1} = 7M$, where $M \in J$, i.e. $15^n = 7M - 6^{2n+1}$.

Required to prove that $15^{n+1} + 6^{2(n+1)+1}$ is divisible by 7.

$$\begin{aligned}15^{n+1} + 6^{2(n+1)+1} &= 15\times 15^n + 6^2\times 6^{2n+1}\\ &= 15\left(7M - 6^{2n+1}\right) + 36\times 6^{2n+1}\\ &= 7\times 15M + 21\times 6^{2n+1}\\ &= 7\left(15M + 3\times 6^{2n+1}\right),\end{aligned}$$ which is divisible by 7.

$\therefore 15^n + 6^{2n+1}$ is divisible by 7 for all $n \geq 0$ by the Principle of Mathematical Induction.

Question 13

(a) $$\begin{aligned}\overrightarrow{BH}.\overrightarrow{CA} &= (\underset{\sim}{h} - \underset{\sim}{b}).(\underset{\sim}{a} - \underset{\sim}{c})\\ &= (\underset{\sim}{a} + \underset{\sim}{c}).(\underset{\sim}{a} - \underset{\sim}{c})\\ &= |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2\\ &= 0, \text{ as } |\underset{\sim}{a}| = |\underset{\sim}{c}| = \text{ radius of the circle}\end{aligned}$$

$\therefore \overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

(b) $$\begin{aligned}V &= \pi\int_0^{\frac{\pi}{2k}}(k+1)^2\sin^2(kx)dx\\ &= \pi(k+1)^2\int_0^{\frac{\pi}{2k}}\frac{1-\cos 2kx}{2}dx\\ &= \frac{\pi(k+1)^2}{2}\left[x - \frac{\sin 2kx}{2k}\right]_0^{\frac{\pi}{2k}}\\ &= \frac{\pi(k+1)^2}{2}\left[\frac{\pi}{2k}\right]\\ &= \frac{\pi^2}{4}\frac{(k+1)^2}{k}.\end{aligned}$$

If $\frac{\pi^2}{4}\frac{(k+1)^2}{k} = \pi^2$ then $(k+1)^2 = 4k$.

$k^2 + 2k + 1 = 4k$.

$k^2 - 2k + 1 = 0$.

$(k-1)^2 = 0$.

$\therefore k = 1$.

(c) If $f(x) = \sin x, \forall x$ then $f(x)$ is not monotonic (or 1 to 1),

$\therefore f^{-1}(x)$ does not exist.

$\therefore g(x)$ is not the inverse of $f(x)$.

(d) Let $P(x) = x^3 + bx^2 + cx + d$.

$P'(x) = 3x^2 + 2bx + c$.

By data, $P'(\alpha) + P'(\beta) + P'(\gamma) = 87$,

$\therefore 3\left(\alpha^2 + \beta^2 + \gamma^2\right) + 2b\left(\alpha + \beta + \gamma\right) + 3c = 87$

$3\times 85 + 2b\left(-b\right) + 3c = 87$

$2b^2 - 3c = 168.$ (1)

But $$\begin{aligned}b^2 &= (-b)^2\\ &= \left(\alpha + \beta + \gamma\right)^2\\ &= \alpha^2 + \beta^2 + \gamma^2 + 2\left(\alpha\beta + \alpha\gamma + \beta\gamma\right)\\ &= 85 + 2c.\end{aligned}$$

$\therefore$ (1) becomes $2\left(85 + 2c\right) - 3c = 168$.

$\therefore c = 168 - 170 = -2$.

$\therefore \alpha\beta + \alpha\gamma + \beta\gamma = -2$.

(d) (i) $\mu = np = 16 \times 0.2 = 3.2$, where p is the probability that a chocolate bar weighs less than 150 g.

$\sigma^2 = npq = 16 \times 0.2 \times 0.8 = 2.56$

$\Pr(x \geq 8) = \Pr\left(z \geq \dfrac{8-3.2}{\sqrt{2.56}}\right)$

$= \Pr(z \geq 3)$

$= 1 - 0.9987$, using the table on page 18 of the question booklet, $\Pr(z \leq 3) = 0.9987$

$= 0.0013$.

(ii) $npq = 2.56$ is too small. As a rule of thumb, the size n must satisfy $npq \geq 10$.

Question 14

(a) $(x-2)\dfrac{dy}{dx} = xy$

$\dfrac{1}{y}dy = \dfrac{x}{x-2}dx$

$= \left(1 - \dfrac{2}{2-x}\right)dx$

$\ln y = x + 2\ln(2-x) + C$

Substituting (0,1) gives $\ln 1 = 0 + 2\ln 2 + C, \therefore C = -\ln 4$.

$\therefore \ln y = x + 2\ln(2-x) - \ln 4$

$= \ln e^x + \ln(2-x)^2 - \ln 4$

$= \ln\left(\dfrac{(2-x)^2 e^x}{4}\right)$.

$\therefore y = \dfrac{(2-x)^2 e^x}{4}$.

(b) The projection of $\underset{\sim}{u}$ on $\underset{\sim}{v}$ has magnitude $\dfrac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{v}|^2}$.

$\underset{\sim}{p} = \dfrac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v} = \lambda_0 \underset{\sim}{v}, \therefore \lambda_0 = \dfrac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{v}|^2}$.

$|\underset{\sim}{u} - \lambda\underset{\sim}{v}|^2 = (\underset{\sim}{u} - \lambda\underset{\sim}{v}).(\underset{\sim}{u} - \lambda\underset{\sim}{v})$

$= |\underset{\sim}{v}|^2 \lambda^2 - 2\underset{\sim}{u}.\underset{\sim}{v}\lambda + |\underset{\sim}{u}|^2$.

The RHS is a quadratic in terms of λ, $\therefore$ it is minimum when $\lambda = -\dfrac{b}{2a} = \dfrac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{v}|^2} = \lambda_0$.

$\therefore |\underset{\sim}{u} - \lambda_0\underset{\sim}{v}|^2 \leq |\underset{\sim}{u} - \lambda\underset{\sim}{v}|^2$.

$\therefore |\underset{\sim}{u} - \lambda_0\underset{\sim}{v}| \leq |\underset{\sim}{u} - \lambda\underset{\sim}{v}|$.

(c) $2ut\sin\theta - \dfrac{g}{2}t^2 = 0$ gives $t = 0$ or $\dfrac{4u\sin\theta}{g}$. $\therefore$ The player hits the target when $t = \dfrac{4u\sin\theta}{g}$,

$x_P = 2ut\cos\theta = \dfrac{8u^2\sin\theta\cos\theta}{g} = \dfrac{4u^2\sin 2\theta}{g}$.

$\therefore \text{Max}(x_P) = \dfrac{4u^2}{g}$, which occurs when $\sin 2\theta = 1$.

$x_T = d + ut = d + \dfrac{4u^2\sin\theta}{g}$.

For the player to hit the target, $x_T \leq x_p$,

$d + \dfrac{4u^2\sin\theta}{g} \leq \dfrac{4u^2\sin 2\theta}{g}$

$d \leq \dfrac{4u^2}{g}(\sin 2\theta - \sin\theta)$.

Let $f(\theta) = \sin 2\theta - \sin\theta$.

$f'(\theta) = 2\cos 2\theta - \cos\theta$

$= 0$ when $2\cos 2\theta - \cos\theta = 0$.

$2\left(2\cos^2\theta - 1\right) - \cos\theta = 0.$

$4\cos^2\theta - \cos\theta - 2 = 0.$

$\cos\theta = \dfrac{1 \pm \sqrt{1+32}}{8} = 0.84 \text{ or } -0.59.$

Given $0 < \theta < \dfrac{\pi}{2}$, clearly $\cos\theta = 0.84$ corresponds to the maximum possible range of the projectile.

When $\cos\theta = 0.84, \sin\theta = \sqrt{1-0.84^2} = 0.54$ and

$\sin 2\theta = 2\sin\theta\cos\theta = 0.91,$

$f(\theta) = 0.91 - 0.54 = 0.37.$

$\therefore d \le 0.37\dfrac{4u^2}{g} = 0.37 \text{ Max}(x_P).$

(d) $\mu = np = 0.95n,$ where p is Pr(not missing a flight)

$\sigma^2 = npq = 0.05 \times 0.95n = 0.0475n$

$\Pr(X > 350) = \Pr\left(z > \dfrac{350 - 0.95n}{\sqrt{0.0475n}}\right) \le 0.01.$

$\therefore \Pr\left(z \le \dfrac{350 - 0.95n}{\sqrt{0.0475n}}\right) \ge 0.99$

From the table on page 18 of the question booklet,

$\Pr(z \le 2.33) = 0.99,$

$\therefore \dfrac{350 - 0.95n}{\sqrt{0.0475n}} = 2.33.$ (1)

$350 - 0.95n = 2.33\sqrt{0.0475n}$

$\left(350 - 0.95n\right)^2 = 2.33^2 \times 0.0475n$

$0.9025n^2 - 665.258n + 122500 = 0$

$n = 358 \text{ or } 379.$

Noting that, from (1), $350 - 0.95n > 0$ for $n = 358$, but < 0 for $n = 379$. $\therefore$ We reject 379 and take only 358.

$\therefore$ The maximum number of tickets that can be sold is 358.